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GLOBAL AXISYMMETRIC STABILITY ANALYSIS FOR A COMPOSITE SYSTEM OF TWO GRAVITATIONALLY COUPLED SCALE-FREE DISCS

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ABSTRACT

In a composite system of gravitationally coupled stellar and gaseous discs, we perform linear stability analysis for axisymmetric coplanar perturbations using the two-fluid formalism. The background stellar and gaseous discs are taken to be scale-free with all physical variables varying as powers of cylindrical radius r with compatible exponents. The unstable modes set in as neutral modes or stationary perturbation configurations with angular frequency $\omega = 0$. The axisymmetrically stable range is bounded by two marginal stability curves derived for stationary perturbation configurations. By the gravitational coupling between the stellar and the gaseous disc components, one only needs to consider the parameter regime of the stellar disc. There exist two unstable regimes in general: the collapse regime corresponding to large-scale perturbations and the ring-fragmentation regime corresponding to short-wavelength perturbations. The composite system will collapse if it rotates too slowly and will succumb to ring-fragmentation instabilities if it rotates sufficiently fast. The overall stable range for axisymmetric perturbations is determined by a necessary D–criterion involving of the effective Mach number squared D_s^2 (i.e., the square ratio of the stellar disc rotation speed to the stellar velocity dispersion but scaled by a numerical factor). Different mass ratio δ and sound speed ratio η of the gaseous and stellar disc components will alter the overall stability. For spiral galaxies or circumnuclear discs, we further include the dynamical effect of a massive dark matter halo. As examples, astrophysical applications to disc galaxies, proto-stellar discs and circumnuclear discs are discussed.

Subject headings: hydrodynamics — ISM: general — galaxies: kinematics and dynamics — galaxies: spiral — galaxies: structure — waves.

1. INTRODUCTION

Axisymmetric instabilities in models of disc galaxies have been investigated extensively in the last century (e.g., Safronov 1960; Toomre 1964; Binney & Tremaine 1987; Bertin & Lin 1996). For a single disc of either gaseous or stellar content, Safronov (1960) and Toomre (1964) originally introduced a dimensionless Q parameter to determine the local stability condition (i.e. $Q > 1$) against axisymmetric ring-like disturbances in the usual Wentzel-Kramers-Brillouin-Jeffreys (WKBJ) or tight-winding approximation. A more realistic model of a disc galaxy would involve both gas and stars as well as an unseen massive dark matter halo, all interacting among themselves through the mutual gravitation.[1] Many theoretical investigations have been conducted along this track for a composite system of two coupled discs (Lin & Shu 1966; Kato 1972; Jog & Solomon 1984a, b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b). In these earlier treatments, either an approach of combined distribution function and fluid or the formalism of two fluids have been adopted in a WKBJ modal analysis. While these model results were initially derived in various galactic contexts, they can be applied or adapted also to, with proper qualifications, relevant self-gravitating disc systems including accretion discs in general, circumnuclear discs, protostellar discs, planetary discs and so forth.

The local WKBJ or tight-winding approximation has been proven to be a powerful technique in analysing disc wave dynamics. Meanwhile, theorists have long been keenly interested in a class of relatively simple disc models referred to as scale-free discs (Mestel 1963; Zang 1976; Lemos et al. 1991; Lynden-Bell & Lemos 1993; Syer & Tremaine 1996; Evans & Read 1998; Goodman & Evans 1999; Shu et al. 2000; Lou 2002; Lou & Fan 2002; Lou & Shen 2003; Shen & Lou 2003; Lou & Zou 2004; Lou & Wu 2004). Scale-free discs, where all pertinent physical variables (e.g., disc rotation speed, surface mass density, angular speed etc.) scale as powers of cylindrical radius r, have become one effective and simple vehicle to explore disc dynamics. Perhaps, the most familiar case is the so-called singular isothermal discs (SIDs) or Mestel discs with an isothermal equation of state and flat rotation curves (Mestel 1963; Zang 1976; Goodman & Evans 1999; Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004). In contrast to the usual WKBJ approximation for perturbations, perturbations in axisymmetric scale-free discs can be treated, in some cases, globally and exactly without the local restriction (i.e., valid only in the short-wavelength regime). It is therefore possible to derive global properties of perturbations. Using scale-free disc

[1] Magnetic field and cosmic-ray gas component are dynamically important on large scales (Fan & Lou 1996; Lou & Fan 1998a, 2003; Lou & Zou 2004 and references therein) in the galactic gas disc of interstellar medium (ISM) but are not considered here for simplicity.

models, Lemos et al. (1991) and Syer & Tremaine (1996) both studied the axisymmetric stability problem for a single disc and found that instabilities first set in as neutral modes or stationary configurations with angular frequency $\omega = 0$.

The main motivation of this paper is to examine the global axisymmetric stability problem in a composite system of two gravitationally coupled scale-free discs. As a more general extension to the previous two-SID analysis (Lou & Shen 2003; Shen & Lou 2003), we further consider a much broader class of rotation curves as well as the equation of state. This contribution gives an explicit proof that stationary configurations ($\omega = 0$) do mark the marginal stability in the two-fluid system, a cogent supplement to our recent investigation on stationary perturbation configurations (Shen & Lou 2004).

2. TWO-FLUID FORMALISM

As an expedient approximation, we treat both discs as razor-thin discs and use either superscripts or subscripts s and g to indicate physical variables with stellar and gaseous disc associations, respectively. The large-scale coupling between the two discs is primarily caused by the mutual gravitational interaction. In the present formulation of large-scale perturbations, we ignore non-ideal diffusive effects such as viscosity, resistivity and thermal conduction, etc. It is then straightforward to write down the basic coplanar fluid equation set for the stellar disc in cylindrical coordinates $(r,\ \theta,\ z)$ within the $z = 0$ plane, namely

$$\frac{\partial \Sigma^s}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma^s u^s)+\frac{1}{r^2}\frac{\partial}{\partial \theta}(\Sigma^s j^s)=0\ , \tag{1}$$

$$\frac{\partial u^s}{\partial t}+u^s\frac{\partial u^s}{\partial r}+\frac{j^s}{r^2}\frac{\partial u^s}{\partial \theta}-\frac{j^{s2}}{r^3}=-\frac{1}{\Sigma^s}\frac{\partial \Pi^s}{\partial r}-\frac{\partial \phi}{\partial r}\ , \tag{2}$$

$$\frac{\partial j^s}{\partial t}+u^s\frac{\partial j^s}{\partial r}+\frac{j^s}{r^2}\frac{\partial j^s}{\partial \theta}=-\frac{1}{\Sigma^s}\frac{\partial \Pi^s}{\partial \theta}-\frac{\partial \phi}{\partial \theta}\ , \tag{3}$$

where Σ^s is the surface mass density, u^s is the radial bulk flow velocity, j^s is the specific angular momentum about the rotation axis along z–direction, Π^s is the vertically integrated effective (two-dimensional) pressure due to stellar velocity dispersion and ϕ is the *total* gravitational potential. For the gaseous disc, we simply replace superscript or subscript s by g in the above three equations. The coupling of the two sets of fluid equations is due to the gravitational potential through the Poisson integral

$$\phi(r,\theta,t)=\oint d\psi\int_0^\infty \frac{-G\Sigma(r',\psi,t)r'dr'}{[r'^2+r^2-2rr'\cos(\psi-\theta)]^{1/2}}\ , \tag{4}$$

where $\Sigma = \Sigma^s + \Sigma^g$ is the total surface mass density of the composite disc system.

The barotropic equation of state assumes the relation between two-dimensional pressure and surface mass density in the form of

$$\Pi = K\Sigma^n, \tag{5}$$

where coefficients $K \geq 0$ and $n > 0$ are constant. This directly leads to the sound speed[2] a defined by

$$a^2=\frac{d\Pi_0}{d\Sigma_0}=nK\Sigma_0^{n-1}, \tag{6}$$

which scales as $\propto \Sigma_0^{n-1}$. The case of $n = 1$ corresponds to an isothermal sound speed a.

From the basic fluid equations for stellar and gaseous discs above (the latter are not written out explicitly), we may derive the axisymmetric equilibrium background properties (Shen & Lou 2004). We presume that in the equilibrium background of axisymmetry, both rotation curves of the two discs scale as $\propto r^{-\beta}$ and both surface mass densities of the two discs scale as $\propto r^{-\alpha}$ where α and β are two constant exponents and the proportional coefficients are allowed to be different in general. By assuming the same power-law indices for the stellar and gaseous disc rotation curves (or equivalently the surface mass densities), it is possible to consistently construct a global axisymmetric background equilibrium for the composite disc system that meets the requirement of radial force balance at all radii [see equation (2) and the corresponding equation with the superscript s replaced by g] and satisfies the Poisson integral (4) simultaneously.

The scale-free condition requires the following relationship among α, β and n (see Syer & Tremaine 1996 and Shen & Lou 2004), namely

$$\alpha = 1+2\beta \ \text{ and } \ n=\frac{1+4\beta}{1+2\beta}\ . \tag{7}$$

Once the rotation curve is specified, all other physical variables are simultaneously determined.

With the knowledge of computing a gravitational potential arising from an axisymmetric power-law surface mass density for the background rotational equilibrium (Kalnajs 1971; Qian 1992; Syer & Tremaine 1996), we can derive a self-consistent axisymmetric background equilibrium surface mass densities as

$$\Sigma_0^s=\frac{A_s^2(D_s^2+1)}{2\pi G(2\beta\mathcal{P}_0)r^{1+2\beta}(1+\delta)},\ \Sigma_0^g=\frac{A_g^2(D_g^2+1)\delta}{2\pi G(2\beta\mathcal{P}_0)r^{1+2\beta}(1+\delta)}\ , \tag{8}$$

where the coefficient $\mathcal{P}_0$ as a function of β involves Γ–functions

$$\mathcal{P}_0\equiv\frac{\Gamma(-\beta+1/2)\Gamma(\beta)}{2\Gamma(-\beta+1)\Gamma(\beta+1/2)} \tag{9}$$

and parameter $\delta \equiv \Sigma_0^g/\Sigma_0^s$ is the ratio of the surface mass density of the gaseous disc to that of the stellar disc, A and D are two dimensionless parameters. We note that the value of $2\beta\mathcal{P}_0$ falls within $(0,\infty)$ for the prescribed range[3] of $\beta \in (-1/4, 1/2)$ and is equal to 1 when $\beta \to 0$ (i.e., the case of two gravitationally coupled SIDs).

From the radial force balance of the background equilibrium, there also exists a relation such that

$$A_s^2(D_s^2+1)=A_g^2(D_g^2+1)\ , \tag{10}$$

[2] In a stellar disc, the velocity dispersion mimics the sound speed to some extent.

[3] The valid range of $\beta \in (-1/4, 1/2)$ is determined by (1) barotropic index $n > 0$ for warm discs, (2) surface mass density exponent $\alpha < 2$ so that the central point mass does not diverge, and (3) this β range is contained within a wider range of $\beta \in (-1/2, 1/2)$ (for a cold disc system) when the computed force arising from the background potential remains finite (Syer & Tremaine 1996).

where $\eta \equiv A_s^2/A_g^2 = a_s^2/a_g^2$ is another handy dimensionless parameter for sound speed ratio squared. We note that A is actually the reduced[4] effective sound speed [scaled by a factor $(1+2\beta)^{1/2}$] and the parameter $D \equiv \mathcal{V}/A$ where $\mathcal{V}$ is the reduced disc rotation speed, is the effective Mach number [see eqns (11)–(12) later]. Condition (10) is very important in our analysis because the rotations of the two discs are not independent of each other but are dynamically coupled. It suffices to examine the parameter regime in either stellar or gaseous disc. In disc galaxies, the typical velocity dispersion in a stellar disc exceeds the sound speed in a gaseous disc, implying $\eta > 1$ so that inequality $D_g^2 > D_s^2$ holds. Therefore, the physical requirement $D_s^2 > 0$ absolutely guarantees $D_g^2 > 0$ and it suffices to consider the stability problem in terms of $D_s^2 > 0$ together with parameters δ and η for different values [note that $D_g^2 = \eta(D_s^2+1)-1$]. With these explanations, we shall express other equilibrium physical variables in terms of parameters A and D.

The specific angular momenta j_0^s and j_0^g about the z–axis and the sound speeds a_s and a_g in the two coupled equilibrium discs are expressed by

$$j_0^s = A_s D_s r^{1-\beta}\ , \qquad j_0^g = A_g D_g r^{1-\beta}\ , \tag{11}$$

$$\begin{aligned} a_s^2 &= nK_s(\Sigma_0^s)^{n-1} = A_s^2/[(1+2\beta)r^{2\beta}]\ ,\\ a_g^2 &= nK_g(\Sigma_0^g)^{n-1} = A_g^2/[(1+2\beta)r^{2\beta}]\ . \end{aligned} \tag{12}$$

The disc angular rotation speed $\Omega \equiv j_0/r^2$ and the epicyclic frequency $\kappa \equiv [(2\Omega/r)d(r^2\Omega)/dr]^{1/2}$ are similarly expressed in terms of A and D as

$$\begin{aligned} \Omega_s &= A_s D_s r^{-1-\beta}\ , \qquad \kappa_s = [2(1-\beta)]^{1/2}\Omega_s\ ,\\ \Omega_g &= A_g D_g r^{-1-\beta}\ , \qquad \kappa_g = [2(1-\beta)]^{1/2}\Omega_g\ , \end{aligned} \tag{13}$$

with $dj_0/dr = r\kappa^2/(2\Omega)$ to simplify later derivations.

2.1. *Linear Perturbation Equations*

For a composite system of two gravitationally coupled discs in rotational equilibrium with axisymmetry, we introduce small coplanar perturbations denoted by subscript 1 along relevant physical variables. The corresponding linearized perturbation equations can be derived from the basic nonlinear equations (1)–(4) as

$$\begin{aligned} &\frac{\partial\Sigma_1^s}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma_0^s u_1^s) + \Omega_s\frac{\partial\Sigma_1^s}{\partial\theta} + \frac{\Sigma_0^s}{r^2}\frac{\partial j_1^s}{\partial\theta} = 0\ ,\\ &\frac{\partial u_1^s}{\partial t} + \Omega_s\frac{\partial u_1^s}{\partial\theta} - 2\Omega_s\frac{j_1^s}{r} = -\frac{\partial}{\partial r}\left(a_s^2\frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right)\ ,\\ &\frac{\partial j_1^s}{\partial t} + \frac{r\kappa_s^2}{2\Omega_s}u_1^s + \Omega_s\frac{\partial j_1^s}{\partial\theta} = -\frac{\partial}{\partial\theta}\left(a_s^2\frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right) \end{aligned} \tag{14}$$

for the stellar disc as well as their counterparts for the gaseous discs, together with the Poisson integral

$$\phi_1(r,\theta,t) = \oint d\psi\int_0^\infty \frac{-G(\Sigma_1^s+\Sigma_1^g)r'dr'}{\left[r'^2+r^2-2rr'\cos(\psi-\theta)\right]^{1/2}} \tag{15}$$

relating the total gravitational potential perturbation ϕ_1 and the total surface mass density perturbation $\Sigma_1 \equiv \Sigma_1^s + \Sigma_1^g$.

Given a Fourier periodic component in the form of $\exp[i(\omega t - m\theta)]$ for small perturbations in general, we write for coplanar perturbations in the stellar disc

$$\begin{aligned} \Sigma_1^s &= \mu^s(r)\exp[i(\omega t - m\theta)]\ ,\\ u_1^s &= U^s(r)\exp[i(\omega t - m\theta)]\ ,\\ j_1^s &= J^s(r)\exp[i(\omega t - m\theta)]\ , \end{aligned} \tag{16}$$

as well as their counterparts in the gaseous disc, together with the total gravitational potential perturbation

$$\phi_1 = V(r)\exp[i(\omega t - m\theta)]\ , \tag{17}$$

where integer m is taken to be non-negative. For axisymmetric $m = 0$ perturbations, we introduce Fourier decompositions in equations (14) –(15) for the stellar disc to deive

$$\begin{aligned} &i\omega\mu^s + \frac{1}{r}\frac{d}{dr}(r\Sigma_0^s U^s) = 0\ ,\\ &i\omega U^s - 2\Omega_s\frac{J^s}{r} = -\frac{d}{dr}\left(a_s^2\frac{\mu^s}{\Sigma_0^s} + V\right)\ ,\\ &i\omega J^s + \frac{r\kappa_s^2}{2\Omega_s}U^s = 0\ . \end{aligned} \tag{18}$$

We do the same in parallel for the gaseous disc to derive

$$\begin{aligned} &i\omega\mu^g + \frac{1}{r}\frac{d}{dr}(r\Sigma_0^g U^g) = 0\ ,\\ &i\omega U^g - 2\Omega_g\frac{J^g}{r} = -\frac{d}{dr}\left(a_g^2\frac{\mu^g}{\Sigma_0^g} + V\right)\ ,\\ &i\omega J^g + \frac{r\kappa_g^2}{2\Omega_g}U^g = 0\ . \end{aligned} \tag{19}$$

For the total gravitational potential perturbation, we simply have

$$V(r) = \oint d\psi\int_0^\infty \frac{-G(\mu^s+\mu^g)\cos(m\psi)r'dr'}{(r'^2+r^2-2rr'\cos\psi)^{1/2}}\ . \tag{20}$$

Equations (18)–(20) are the basic coplanar perturbation equations used for our axisymmetric stability analysis.

2.2. *Axisymmetric Stability Analysis*

For axisymmetric stability analysis with radial oscillations, we choose the Kalnajs potential-density pairs below because perturbations can be generally expanded in terms of such complete basis functions (Kalnajs 1971; Binney & Tremaine 1987; Lemos et al. 1991; Lou & Shen 2003). Specifically, we take

$$\begin{aligned} &\mu^s = \sigma^s r^{-3/2}\exp(i\xi\ln r)\ , \qquad \mu^g = \sigma^g r^{-3/2}\exp(i\xi\ln r)\ ,\\ &V = -2\pi G r(\mu^s+\mu^g)\mathcal{N}_m(\xi)\ , \end{aligned} \tag{21}$$

where ξ is a 'wavenumber' characterizing the radial variation scale, σ^s and σ^g are two small real coefficients and the parameter function

$$\mathcal{N}_m(\xi) = \frac{\Gamma(m/2+i\xi/2+1/4)\Gamma(m/2-i\xi/2+1/4)}{2\Gamma(m/2+i\xi/2+3/4)\Gamma(m/2-i\xi/2+3/4)} \tag{22}$$

[4] By the adjective "reduced", we refer to the part of a physical variable after removing the power-law radial dependence. For example, in the disc rotation speed $v = \mathcal{V}r^{-\beta}$, quantity $\mathcal{V}$ is referred to as the reduced disc rotation speed.

is the Kalnajs function (Kalnajs 1971) that involves Γ−functions of complex arguments. Note that $\mathcal{N}_m$ is even in ξ. It then suffices to consider only $\xi \geq 0$.

Using the first mass conservations in equations (18) and (19) respectively, we infer $U \propto i\omega r^{1/2+2\beta+i\xi}$ (see also Lou & Zou 2004). By potential-density pair (21), equations (18) and (19) reduce to

$$(\omega^2 - H_1)U^s = -G_2 U^g \ , \qquad (\omega^2 - H_2)U^g = -G_1 U^s \tag{23}$$

in the limit of $\omega \to 0$, where parameter functions H_1, H_2, G_1 and G_2 are explicitly defined by

$$\begin{aligned} H_1 &\equiv \kappa_s^2 + \left(\frac{a_s^2}{r} - 2\pi G\mathcal{N}_0\Sigma_0^s\right)\frac{(\xi^2+1/4)}{r} \ , \\ H_2 &\equiv \kappa_g^2 + \left(\frac{a_g^2}{r} - 2\pi G\mathcal{N}_0\Sigma_0^g\right)\frac{(\xi^2+1/4)}{r} \ , \\ G_1 &\equiv 2\pi G\mathcal{N}_0\Sigma_0^s\frac{(\xi^2+1/4)}{r} > 0 \ , \\ G_2 &\equiv 2\pi G\mathcal{N}_0\Sigma_0^g\frac{(\xi^2+1/4)}{r} > 0 \ . \end{aligned} \tag{24}$$

The axisymmetric dispersion relation in the composite system follows from equation (23)

$$\omega^4 - (H_1+H_2)\omega^2 + (H_1H_2 - G_1G_2) = 0 \tag{25}$$

in the limit of $\omega \to 0$, which is identical in form with earlier results obtained in the WKBJ regime (Jog & Solomon 1984a; Shen & Lou 2003). It would be of interest to note that the conditions for the stellar disc and the gaseous disc to be separately stable are $H_1 > 0$ and $H_2 > 0$, respectively. It is also reminded here that in the familiar WKBJ regime, the dispersion relation in a single disc is

$$\omega^2 = \kappa^2 + k^2a^2 - 2\pi G \mathrm{k}\Sigma_0 \ , \tag{26}$$

where k is the radial wavenumber, whereas in the present global analysis as applied to a single disc, we have in the limit of $\omega \to 0$

$$\omega^2 = \kappa^2 + \frac{(\xi^2+1/4)}{r^2}a_s^2 - 2\pi G\frac{(\xi^2+1/4)\mathcal{N}_0}{r}\Sigma_0 \ . \tag{27}$$

If we replace $\mathcal{N}_0(\xi)$ approximately by $(\xi^2+1/4)^{-1/2}$ in the asymptotic regime of $\xi \gg 1$, we readily identify the correspondence between the effective wavenumber $(\xi^2+1/4)^{1/2}$ and $\mathrm{k}r$ in the WKBJ limit of $\mathrm{k}r \gg 1$ by directly comparing dispersion relations (26) and (27). Physically, dispersion relation (27) is more generally applicable beyond the WKBJ regime and is globally accurate only in the limit of $\omega \to 0$ (Shu et al. 2000).

Getting back to the composite system, there are two real roots[5] of ω^2 of equation (25)

$$\omega_\pm^2 = \frac{1}{2}\{(H_1+H_2) \pm [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ , \tag{28}$$

with ω_+^2 root being always positive[6] (Shen & Lou 2003). For the purpose of axisymmetric stability analysis, we only need to examine ω_-^2 root, namely

$$\omega_-^2 = \frac{1}{2}\{(H_1+H_2) - [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ . \tag{29}$$

As the right-hand side of the above equation (29) is always real, axisymmetric instabilities set in as stationary perturbation configurations with $\omega_-^2 = 0$ and leads to the marginal stability condition

$$H_1H_2 = G_1G_2 \tag{30}$$

that requires inequality $H_1 + H_2 \geq 0$. This inequality can be shown in a straightforward manner to be automatically satisfied if equation (30) holds true. Let us first write

$$H_1 = F_1 - G_1 \qquad \text{and} \qquad H_2 = F_2 - G_2 \ , \tag{31}$$

where F_1 and F_2 are explicitly defined by

$$F_1 \equiv \kappa_s^2 + \frac{(\xi^2+1/4)}{r^2}a_s^2 > 0 \ , \ F_2 \equiv \kappa_g^2 + \frac{(\xi^2+1/4)}{r^2}a_g^2 > 0 \ . \tag{32}$$

It then follows from condition (30) of $H_1H_2 = G_1G_2$ that

$$F_1F_2 = F_1G_2 + F_2G_1 \ . \tag{33}$$

As F_1, F_2, G_1 and G_2 are all positive, we immediately conclude that

$$F_1 - G_1 > 0 \qquad \text{and} \qquad F_2 - G_2 > 0 \ , \tag{34}$$

which finally leads to

$$H_1 + H_2 > 0 \ . \tag{35}$$

The composite disc system becomes inevitably unstable for $H_1 + H_2 < 0$ because of $\omega_-^2 < 0$. Else if $H_1 + H_2 \geq 0$ but with $H_1H_2 - G_1G_2 < 0$, we again have $\omega_-^2 < 0$ for instabilities. Only when $H_1 + H_2 \geq 0$ and $H_1H_2 - G_1G_2 > 0$ at the same time can the composite disc system be stable against axisymmetric coplanar perturbations. This is an important necessary stability criterion for a composite system of two gravitationally coupled discs. In other words, once one disc is unstable by itself (i.e., $H_1 < 0$ or $H_2 < 0$ or both), the two-disc system must be unstable; even if the two discs are both separately stable, the composite disc system can still become unstable (i.e., $H_1 > 0$ and $H_2 > 0$ but $H_1H_2 - G_1G_2 < 0$).

By inserting expressions of H_1, H_2, G_1 and G_2 into the marginal stability condition (30) together with requirements of the background rotational equilibrium, we readily derive a quadratic equation in terms of $y \equiv D_s^2$, namely

$$C_2y^2 + C_1y + C_0 = 0 \ , \tag{36}$$

where the coefficients are explicitly defined by

$$\begin{aligned} C_2 &\equiv \mathcal{B}_0\mathcal{H}_0\eta \ , \\ C_1 &\equiv \left[(\mathcal{B}_0 - \mathcal{A}_0)\mathcal{H}_0 + \frac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0-\mathcal{B}_0)}{(1+\delta)}\right]\eta \\ &\quad - \frac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0+\mathcal{B}_0\delta)}{(1+\delta)} \ , \\ C_0 &\equiv \left[-\mathcal{A}_0\mathcal{H}_0 + \frac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0-\mathcal{B}_0)}{(1+\delta)}\right]\eta \\ &\quad + (\mathcal{A}_0+\mathcal{B}_0)^2 - \frac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0+\mathcal{B}_0\delta)}{(1+\delta)} \ , \end{aligned} \tag{37}$$

[5] One can show that the determinant of equation (25) $\Delta \equiv (H_1 - H_2)^2 + 4G_1G_2 > 0$ is always true.

[6] If $H_1 + H_2 \geq 0$, then $\omega_+^2 > 0$; otherwise if $H_1 + H_2 < 0$, then at least one of H_1 and H_2 is negative. It therefore follows that $H_1H_2 - G_1G_2 < 0$ and hence $\omega_+^2 > 0$.

and

$$
\begin{aligned}
&\mathcal{A}_0(\xi) \equiv \xi^2 + 1/4 \ , \\
&\mathcal{B}_0(\beta) \equiv (1+2\beta)(-2+2\beta) \ , \\
&\mathcal{C}(\beta) \equiv (1+2\beta)/(2\beta\mathcal{P}_0) \ , \\
&\mathcal{H}_0(\beta,\xi) \equiv \mathcal{C}\mathcal{N}_0\mathcal{A}_0 + \mathcal{B}_0 \ .
\end{aligned} \tag{38}
$$

This quadratic equation (36) of $y \equiv D_s^2$ can be readily proven to always have two real solutions (Shen & Lou 2004). Only the positive portion of D_s^2 solutions can be regarded as physically acceptable. Typically, there exist two different regimes bounded by the marginal D_s^2 stability curves that are unstable against axisymmetric perturbations, that is, the collapse regime for long-wavelength perturbations and the ring-fragmentation regime for short-wavelength perturbations. In contrast to the short-wavelength WKBJ approximation, the collapse regime is novel and exact. Systems with too fast a rotation parameter D_s^2 will fall into the ring-fragmentation regime (Safronov 1960; Toomre 1964; Syer & Tremaine 1996; Lou & Fan 1998a, b; Shu et al. 2000; Lou 2002; Lou & Shen 2003, 2004; Shen & Lou 2003), while those with too slow a D_s^2 parameter will fall into the collapse regime. Shown in Fig. 1 is an example of illustration with $\beta = 1/4$, $\eta = 1$ and an unconstrained δ. We note that cases with $\eta = 1$ are essentially the same as those of a single disc (Shen & Lou 2004). The boundaries of the two regimes of instabilities shown in Fig. 1 vary with different parameters $\eta > 1$ and δ for chosen values of β that describes the entire scale-free radial profile of a composite disc system (i.e., disc rotation curves, surface mass densities and barotropic equation of state). Qualitatively, the increase of either η and δ will aggravate the ring-fragmentation instability while suppress the large-scale collapse instability (Shen & Lou 2003). While this can be directly seen from equation (36) in the relevant parameter regimes, it can also be understood physically in terms of the dynamical coupling between the two discs through condition (10) constrained by scale-free conditions. For a larger η, the reduced gas disc rotation speed $\mathcal{V}_g$ will exceed the reduced stellar disc rotation speed $\mathcal{V}_s$ by a larger margin and this tend to prevent an overall collapse of a composite disc system. In other words, a gaseous disc component with a relatively lower sound speed seems to prevent a collapse (Shen & Lou 2003). More details and examples can be found in Shen & Lou (2004).

It is well known that a Q parameter can be defined to determine local axisymmetric stability of a single-disc system (Safronov 1960; Toomre 1964; Binney & Tremaine 1987). For a composite disc system, it has been attempted to introduce an effective $Q_{\rm eff}$ parameter (Elmegreen 1995; Jog 1996; Lou & Fan 1998b). As a result of straightforward numerical computations, we have recently introduced a powerful D–criterion for axisymmetric stability of a composite system of two coupled SIDs (Shen & Lou 2003). It is natural to further generalize this D–criterion for axisymmetric stability of a composite system of two coupled barotropic discs by straightforward numerical computations as shown in Fig. 1. In the present case, it is more practical and simple to use D_s^2 parameter.

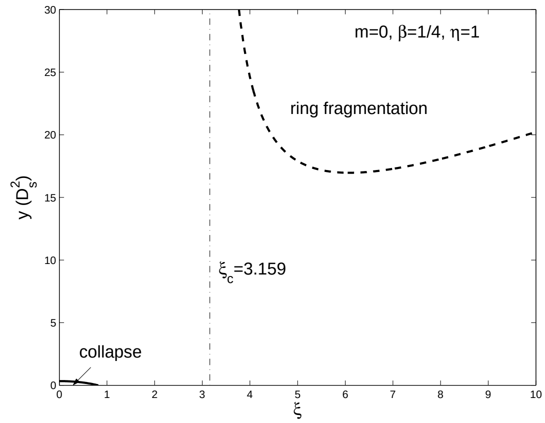

FIG. 1.— One illustrating example for the two unstable regimes with $m = 0$, $\beta = 1/4$ and $\eta = 1$. The collapse regime is at the lower-left corner, while the ring-fragmentation regime is at the upper-right corner. In this special case of $\eta = 1$, parameter δ can be arbitrary as can be seen from equation (37), namely, when $\eta = 1$ the coefficients C_2, C_1 and C_0 turn out to be independent of δ. The vertical dash-dotted line is the location of ξ_c where $\mathcal{H}_0 = 0$ and C_2 vanishes. The solid line and the dashed line bound the collapse regime and the ring fragmentation regime, respectively. Only when D_s^2 falls within the range between the top of the collapse regime and the bottom of the ring fragmentation regime can a composite disc system become stable against all axisymmetric coplanar perturbations.

2.3. *Partial Disc Systems and Applications to Disc Galaxies*

From observations of more or less flat rotation curves of most disc galaxies, massive dark matter halos have been inferred to exist ubiquitously as long as the Newtonian gravity remains valid on galactic scales. If we naively attempt to relate theoretical results obtained in Section 2.2 to a typical disc galaxy, we may take the simple isothermal equation of state as an example of illustration. The relevant parameters for a composite SID system are then chosen as $\beta = 0$ and $a_s = 50$ km s^{-1}, $\mathcal{V}_s = 220$ km s^{-1}, $\delta = 0.1$ and $\eta = 50$. Unfortunately, such a composite system of two coupled SIDs is inevitably unstable against ring-fragmentation perturbations because of a $D_s^2 \simeq 20$, far exceeding the maximum value necessary for being stable against ring fragmentations. This dilemma can be resolved by attributing to an additional gravitational potential associated with an unseen massive dark matter halo. We refer to a composite disc system in association with an axisymmetric dark matter halo as a composite system of *partial discs* (e.g. Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Fan 2002; Shen & Lou 2003, 2004). In a simple treatment, the dynamical effect of a dark matter halo is modeled as only to contribute an axisymmetric gravitational potential Φ in the background rotational equilibrium but not to respond to coplanar perturbations in the composite disc system. With $\phi_T \equiv \Phi + \phi$, we conveniently introduce a dimensionless parameter $\mathcal{F} \equiv \phi/\phi_T$ as the ratio of the potential arising from the composite disc system to that of the entire system including the presumed axisymmetric dark matter halo. The full-disc system corresponds to $\mathcal{F} = 1$ (i.e. $\Phi = 0$) and the partial-disc system corresponds to $0 < \mathcal{F} < 1$ (i.e., $\Phi \neq 0$).

For a composite system of two gravitationally coupled partial discs, we follow the same procedure of analysing coplanar perturbations in full discs to derive a similar quadratic equation of D_s^2 as the stationary dispersion relation. In a nutshell, we can simply replace all $\mathcal{N}_0(\xi)$ in our theoretical results by $\mathcal{F}\mathcal{N}_0(\xi)$ to accomplish this generalization or extension. The introduction of the ratio parameter $\mathcal{F}$ will significantly reduce both the ring-fragmentation regime and the collapse regime, as already can be seen from a comparative study of the WKBJ and global approaches (Shen & Lou 2003).

For the purpose of illustrating the stabilizing effect of a partial-disc system, we simply take $\mathcal{F} = 0.1$ with other parameters used earlier in this section. The minimum value of D_s^2 for unstable ring fragmentations now becomes ~ 650, far beyond the actual value of $D_s^2 \simeq 20$ in a disc galaxy. Meanwhile, the collapse regime completely disappears. Therefore, a typical composite system of two coupled partial discd is fairly stable against axisymmetric coplanar perturbations.

3. DISCUSSION AND SUMMARY

The main thrust of this investigation is to model linear coplanar perturbations of axisymmetry ($m = 0$) in a composite system of two-fluid scale-free discs with one intended for a stellar disc and the other intended for a gaseous disc. The two discs are dynamically coupled through the mutual gravitational interaction. In order to include the dynamical effect of a massive dark matter halo with axisymmetry, we further describe a composite system of two coupled partial discs (e.g. Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004; Lou & Wu 2004). In a global perturbation analysis, we show that axisymmetric instabilities set in as stationary perturbation configurations with $\omega = 0$. The marginal D_s^2 stability curves (characterized by the stationary configurations) delineate two different unstable regimes, namely, the collapse regime for large-scale perturbations and the ring-fragmentation regime for short-wavelength perturbations. Apparently, the composite disc system becomes less stable than a single-disc system and can be unstable with the two discs being stable separately (Lou & Fan 1998b). In our analysis, stationary perturbation configurations turn out to be more than just an alternative equilibrium state, especially in view of the stability properties.

The basic results of this paper are generally applicable to self-gravitating disc systems with or without axisymmetric dark matter halos. The two-fluid treatment contains more realistic elements than a single-disc formulation in the context of disc galaxies. In addition to astrophysical applications to disc galaxies, the studies presented here can be valuable for exploring the dynamical evolution of protostellar discs and circumnuclear discs.

In the context of a proto-stellar disc, it is the usual case to ignore the self-gravity effect. By considering the self-gravity of a composite disc system, our analysis indicates several qualitative yet interesting results. For example, if the initial disc system rotates sufficiently fast, the ring fragmentation (see the upper-right part of Fig. 1) can occur at relatively small radial scales. By further non-axisymmetric fragmentations, these condensed rings of materials may eventually become birthplaces of planets. On the other hand, if the initial disc system rotates sufficiently slow, then gravitational collapse can be induced by perturbations of relatively large radial scales (see the lower left corner of Fig. 1). Once such a perturbation develops in the background equilibrium disc, it grows rapidly and destabilizes the disc. Subsequently, the system undergoes global Jeans collapse to form a central young stellar object. Finally, if the initial disc system rotates in a regime stable against all axisymmetric perturbations (see Fig. 1), there might be two possibilities: (1) the composite disc system might become unstable caused by non-axisymmetric perturbations (not analyzed here) and (2) the disc rotation may be gradually slowed down by some braking mechanisms (e.g. magnetic field not included here and outflows or winds) and the disc eventually succumbs to a central collapse induced by large-scale perturbations.

Likewise, in the context of a circumnuclear disc around the center of a galaxy, we can readily conceive similar physical processes in parallel. One important distinction is that a dark-matter halo should play an important dynamical role so that a formulation of a partial composite disc system would be more appropriate. Here, the ring fragmentation can be induced by relatively small-scale perturbations in a disc system of sufficiently fast rotation. Such a ring of relatively dense materials around the galactic center would be a natural birthplace for circumnuclear starburst activities (e.g. Lou et al. 2001). Depending on the evolution history of a circumnuclear disc system, it may be stable initially and gradually lose angular momentum by generating and damping spiral magnetohydrodynamic (MHD) density waves (Lou et al. 2001). When the disc rotation becomes sufficiently slow, Jeans collapse induced by large-scale perturbations can set in to form a bulge or a supermassive black hole.

In summary, our global analysis shows the possible presence of an evolution stage for a composite disc system against all axisymmetric coplanar perturbations. More importantly, we reveal the parameter regime of ring fragmentation and the parameter regime of large-scale collapse. Astrophysical applications are discussed in the contexts of disc galaxies, proto-stellar discs and circumnuclear disks.

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